ISRAEL JOURNAL OF MATHEMATICS, Vol. 31, No. 2, 1978

LOWER BOUNDS AT INFINITY OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS IN THE EXTERIOR OF A PROPER CONE

BY
MINORU MURATA AND YOSHIHIRO SHIBATA

ABSTRACT

An extension of a classical theorem of Rellich to the exterior of a closed proper convex cone is proved: Let Γ be a closed convex proper cone in $\mathbf{R}^n$ and $-\Gamma'$ be the antipodes of the dual cone of Γ. Let $P = P(-i\partial/\partial x) = Q(-i\partial/\partial x)\Pi_{j=1}^{k} P_j(-i\partial/\partial x)^{m_j}$ be a partial differential operator with constant coefficients in $\mathbf{R}^n$, where $Q(\zeta) \neq 0$ on $\mathbf{R}^n - i\Gamma'$ and P_j is an irreducible polynomial with real coefficients. Assume that the closure of each connected component of the set $\{\zeta \in \mathbf{R}^n - i\Gamma';\ P_j(\zeta) = 0,\ \text{grad}\ P_j(\zeta) \neq 0\}$ contains some real point on which grad $P_j \neq 0$ and grad $P_j \notin \Gamma \cup (-\Gamma)$. Let C be an open cone in $\mathbf{R}^n - \Gamma$ containing both normal directions at some such point, and intersecting each normal plane of every manifold contained in $\{\xi \in \mathbf{R}^n;\ P(\xi) = 0\}$. If $u \in \mathcal{S}' \cap L^2_{\text{loc}}(\mathbf{R}^n - \Gamma)$ and the support of $P(-i\partial/\partial x)u$ is contained in Γ, then the condition

$$\varliminf_{R\to\infty} R^{-1} \int_{C_R} |u(x)|^2 dx = 0, \qquad C_R = \{x \in C;\ R < |x| < 2R\}$$

implies that the support of u is contained in Γ.

1. Introduction

Let us consider the equation

$$P(x, -i\partial/\partial x)u = 0 \qquad \text{in } \Omega \tag{1.1}$$

where Ω is an unbounded domain in $\mathbf{R}^n$. When Ω is the exterior of a compact set, Rellich [10] proved that a solution of the equation $\Delta u + u = 0$ in Ω must vanish identically if $u(x)|x|^{(n-1)/2} \to 0$ as $x \to \infty$. Extensions of this result have been given for large classes of operators (see [1,2], [4], [5], [7], [8,9], [14]). In particular, Hörmander [4] and Murata [8,9] independently completed the study of the constant coefficients case. When the boundary of Ω extends to infinity,

Received March 23, 1978

however, results which have been published are incomplete. Konno [6] gave lower bounds for solutions of Schrödinger-type equations when Ω is the exterior of an elliptic paraboloid. Agmon [1] treated the Schrödinger-type equations in the case that Ω contains a half space and the potential satisfies proper conditions which are not fulfilled by a constant function. Shibata [12] considered partial differential equations with constant coefficients in a half space, and gave lower bounds for their solutions which satisfy suitable zero boundary conditions on the whole boundary. As for Schrödinger-type equations in a domain which is contained in a half space, related results were given for solutions with Dirichlet zero conditions (see, [1], [10], [13]).

Our aim in this paper is to give lower bounds for solutions of (1.1) when Ω is the exterior of a closed convex proper cone and P is a differential operator with constant coefficients. Our results can be applied to the reduced wave equation, and are new even in this case.

Now, we explain notations in order to state the results. $\mathbf{R}^n$ denotes n-dimensional Euclidean space and points of $\mathbf{R}^n$ are written as $x = (x_1, \cdots, x_n)$. For differentiation we use the symbol $D = -i(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. For the real dual space of $\mathbf{R}^n$ we use the same notation $\mathbf{R}^n$ and their points are written as $\xi = (\xi_1, \cdots, \xi_n)$. $\mathbf{C}^n$ denotes n-dimensional unitary space and points of $\mathbf{C}^n$ are written as $\zeta = (\zeta_1, \cdots, \zeta_n)$ $(\zeta_j \in \mathbf{C})$ or $\xi + i\eta$ with $\xi, \eta \in \mathbf{R}^n$. In this way $\mathbf{C}^n$ is regarded as the cartesian product of two copies of $\mathbf{R}^n$. When $A, B \subset \mathbf{R}^n$, $A + iB$ is the set of all $\zeta = \xi + i\eta$ with $\xi \in A$, $\eta \in B$. For any subset A of $\mathbf{R}^n$, we denote by $-A$ the set $\{x \in \mathbf{R}^n;\ -x \in A\}$. Let Γ be a closed convex proper cone with its vertex at the origin, Γ' be its dual cone, that is, $\Gamma' = \{\xi \in \mathbf{R}^n;\ x \cdot \xi > 0$ for any $x \in \Gamma \backslash \{0\}\}$, and $\Gamma_y = y + \Gamma$ for any $y \in \mathbf{R}^n$. Then we have the following theorem.

THEOREM 1. *Let Γ be a closed convex proper cone in $\mathbf{R}^n$ with its vertex at the origin and $P(\zeta) = Q(\zeta)\prod_{j=1}^{p}(P_j(\zeta))^{m_j}$, where $Q(\zeta)$ is a polynomial with complex coefficients such that $Q(\zeta) \neq 0$ for any $\zeta \in \mathbf{R}^n - i\Gamma'$ and $P_j(\zeta)$ $(j = 1, \cdots, p)$ are irreducible polynomials with real coefficients. Set*

$$A_j = \{\zeta \in \mathbf{R}^n - i\Gamma';\ P_j(\zeta) = 0,\ \operatorname{grad} P_j(\zeta) \neq 0\} = \bigcup_k A_j^k,$$

$$B_j = \{\xi \in \mathbf{R}^n;\ P_j(\xi) = 0,\ \operatorname{grad} P_j(\xi) \neq 0,\ \operatorname{grad} P_j(\xi) \notin \Gamma \cup (-\Gamma)\}$$

where A_j^k is a connected component of A_j. Assume that the closure $\overline{A_j^k}$ of each A_j^k intersects B_j. Let y be a point in $\mathbf{R}^n$. Let C be an open cone in $\mathbf{R}^n - \Gamma_y$ such that (i) *for some $\xi^{jk} \in \overline{A_j^k} \cap B_j$, $C \ni \pm \operatorname{grad} P_j(\xi^{jk})$;* (ii) *for every real analytic manifold $M \subset \{\xi \in \mathbf{R}^n;\ P(\xi) = 0\}$ and $\xi^0 \in M$, C contains some normal of M at ξ^0. Set*

$$C_R = \{x \in C;\ R < |x| < 2R\}.$$

If $u \in \mathcal{S}' \cap L^2_{\mathrm{loc}}(\mathbf{R}^n - \Gamma_y)$ satisfies the conditions

(1.2) $$P(D)u = f, \qquad \operatorname{supp} f \subset \Gamma_y,$$

(1.3) $$\varliminf_{R\to\infty} R^{-1}\int_{C_R} |u(x)|^2 dx = 0,$$

then $\operatorname{supp} u \subset \Gamma_y$.

The remainder of this paper is organized as follows. In section 2 the proof of Theorem 1 is given. In section 3 we examine the assumptions on the geometry of $P(\zeta)$ and give some examples. Moreover, we state that Theorem 1 can be generalized to more general equations $(P(D)+\Sigma_{j=1}^{N} q_j(x)Q_j(D))u = f$.

2. Proof of Theorem 1

First, we arrange some results which are used to prove Theorem 1. The following proposition is derived immediately from *the edge-of-wedge theorem* (see [11, theorem B] and [15, section 27]).

PROPOSITION 2.1. *Let S be an open cone in $\mathbf{R}^n$ and V be the intersection of S with some bounded open ball with center at the origin of $\mathbf{R}^n$. Let E be a non-empty open set in $\mathbf{R}^n$. Set*

$$W = E + iV.$$

If f is holomorphic in W, and

$$\lim_{\substack{\eta\to 0\\ \eta\in V}} \int_E f(\xi+i\eta)\phi(\xi)d\xi = 0$$

for every infinitely differentiable function ϕ with compact support in E, it follows that $f \equiv 0$.

The following are well-known results of functions that are holomorphic in a tubular domain (see [15, section 25]). Let $g(\xi+i\eta)$ be holomorphic in a tubular domain $\mathbf{R}^n - i\Gamma'$. We shall use the term *spectral function* of the function $g(\xi+i\eta)$ to denote the distribution $f \in \mathcal{D}'$ possessing the following properties:

(a) $f(x)e^{x\cdot\eta} \in \mathcal{S}'$ *for all* $\eta \in -\Gamma'$,

(b) $g(\xi+i\eta) =$ *the Fourier transform of* $f(x)e^{x\cdot\eta}$

$$= \int e^{-i(\xi+i\eta)\cdot x} f(x)dx \qquad \text{for all } \xi+i\eta \in \mathbf{R}^n - i\Gamma'.$$

Here, the function $g(\xi+i\eta)$ will be called the *Fourier–Laplace transform* of the spectral function $f(x)$, and be denoted by $\hat{f}(\xi+i\eta)$. We shall call a cone S a *compact subcone* of Γ' if the intersection of $\bar{S}$ and the unit ball is contained in Γ'.

PROPOSITION 2.2. (1) *For the support of $f \in \mathcal{S}'(\mathbf{R}^n)$ to be contained in Γ, it is necessary and sufficient that $\hat{f}(\xi + i\eta)$ is holomorphic in $\mathbf{R}^n - i\Gamma'$ and that for any compact subcone C of Γ' there exist positive constants α, β and $M(C)$ such that α and β are independent of C and*

$$(2.1)\quad |\hat{f}(\xi + i\eta)| \leqq M(C)(1 + |\xi + i\eta|^{\alpha})(1 + |\eta|^{-\beta}), \qquad \xi + i\eta \in \mathbf{R}^n - iC.$$

(2) *If $\hat{f}(\xi + i\eta)$ is holomorphic in $\mathbf{R}^n - i\Gamma'$ and satisfies* (2.1), *then there exists in $\mathcal{S}'$ a unique boundary value*

$$\hat{f}(\xi) = \lim_{\substack{\eta \to 0 \\ \eta \in -\Gamma'}} \hat{f}(\xi + i\eta) \in \mathcal{S}',$$

that is independent of the sequence $\eta \to 0$ for $\eta \in -\Gamma'$, and the spectral function $f(x)$ of $\hat{f}(\xi + i\eta)$ is equal to the inverse Fourier transform of $\hat{f}(\xi)$.

The following is the well-known inequality due to Malgrange (see [3], [15]).

PROPOSITION 2.3. *Let U be an open set in $\mathbf{C}^n$, $F(\zeta)$ a holomorphic function in U, and $P(\zeta)$ a polynomial of degree $\leqq m$. Let $\Psi(\zeta)$ be a non-negative integrable function with compact support contained in U, depending only on $|\zeta_1|, \cdots, |\zeta_n|$. Then*

$$(2.2)\qquad |F(0)P^{(\alpha)}(0)| \int |\zeta^{\alpha}|\Psi(\zeta)d\zeta \leqq M_{m,|\alpha|} \int |F(\zeta)P(\zeta)|\Psi(\zeta)d\zeta,$$

where $d\zeta$ is the Lebesgue measure in $\mathbf{C}^n$ and $M_{m,|\alpha|}$ is a positive constant depending only on m and $|\alpha|$.

Now, let us prove Theorem 1. We may assume that $y = 0$. Under the assumptions of Theorem 1, we shall show that $\hat{f}(\zeta)/P(\zeta)$ is holomorphic in $\mathbf{R}^n - i\Gamma'$. Without loss of generality, we may assume that $\Gamma \ni (x_1, 0, \cdots, 0)$ $(x_1 > 0)$ and $\partial P_j/\partial \xi_n(\xi^{jk}) \neq 0$. Hence ξ^{jk} is the point in $\overline{A_j^k} \cap B_j$ with $\pm \operatorname{grad} P_j(\xi^{jk}) \in C$. By the implicit function theorem, the root of P_j near ξ^{jk} is of the form

$$\zeta_n = s(\zeta'), \qquad \zeta' = \xi' + i\eta' \in \Omega$$

where s is a holomorphic function in an open set $\Omega \subset \mathbf{C}^{n-1}$, $s(\xi')$ is real valued for $\xi' \in \Omega \cap \mathbf{R}^{n-1}$, $\xi^{jk} = ((\xi^{jk})', \ s((\xi^{jk})'))$ and $\operatorname{grad} P_j(\xi^{jk})$ is proportional to $(\operatorname{grad} s((\xi^{jk})'), -1) \notin \Gamma \cup (-\Gamma)$. By Taylor expansion, we have $\operatorname{Im} s(\xi' + i\eta') = s'(\xi') \cdot \eta' + O(|\eta'|^2)$, where $s' = \operatorname{grad} s$. Since $(\eta', s'(\xi') \cdot \eta')$ is proportional to the tangent plane of $\{(\xi', s(\xi')), \xi' \in \Omega \cap \mathbf{R}^{n-1}\}$ at $(\xi', s(\xi'))$ and $(s'((\xi^{jk})'), -1) \notin \Gamma \cup (-\Gamma)$, there exist a small compact subcone S of $\{\eta' \in \mathbf{R}^{n-1}; (\eta', s'((\xi^{jk})') \cdot \eta') \in -\Gamma'\}$, a small open ball B with center at the origin of $\mathbf{R}^{n-1}$, and a neighborhood E of $(\xi^{jk})'$ such that the set $\{(\xi' + i\eta, s(\xi' + i\eta')); \xi' \in E, \eta' \in V = S \cap B\}$ is contained in $\mathbf{R}^n - i\Gamma'$. We shall show

$$\lim_{\substack{\eta'\to 0\\ \eta'\in V}} \int_E (\partial/\partial\zeta_n)^\nu \hat{f}(\xi'+i\eta', s(\xi'+i\eta'))\phi(\xi')d\xi' = 0 \tag{2.3}$$

for any $\phi(\xi')\in C_0^\infty(E)$ and $0\leqq \nu \leqq m_j-1$. If we show (2.3), it follows from Proposition 2.1 that $\hat{f}$ vanishes of order m_j at $\{(\xi'+i\eta', s(\xi'+i\eta'));\ \xi'+i\eta'\in E+iV\}$. This and the assumption that $\overline{A_j^k}\cap B_j \ni \xi^{jk}$ imply that $\hat{f}/(P_j)^{m_j}$ is holomorphic in $\mathbf{R}^n - i\Gamma'$ (see [15, lemma 8.4]). We now repeat the argument, letting $\hat{f}/(P_j)^{m_j}$ and $P_{j'}$ take the place of $\hat{f}$ and P_j, respectively. Clearly this argument can be repeated any number of times. So we have that $\hat{f}/P$ is holomorphic in $\mathbf{R}^n - i\Gamma'$.

Let us show (2.3). For any $\phi\in C_0^\infty(E)$ and $\eta'\in V\cup\{0\}$, we set

$$\Phi_{\eta'}(x) = \int e^{-i\{(\xi'+i\eta')x'+x_n s(\xi'+i\eta')\}}\phi(\xi')d\xi', \quad \Phi(x)=\Phi_0(x).$$

Choose a small open conic neighborhood $\tilde{\Gamma}$ of $\Gamma-\{0\}$ such that

$$\sum_{j=1}^{n-1} |x_j + x_n\partial s/\partial\xi_j(\xi'+i\eta')|^2 \neq 0, \qquad x\in\tilde{\Gamma},\quad \xi'+i\eta'\in E+i(V\cup\{0\}).$$

Putting

$$L = \sum_{j=1}^{n-1}\left(\sum_{l=1}^{n-1}|x_l + x_n\partial s/\partial\xi_l(\xi'+i\eta')|^2\right)^{-1}\overline{(x_j + x_n\partial s/\partial\xi_j(\xi'+i\eta))}(-i\partial/\partial\xi_j),$$

we have for any non-negative integer k

$$(L)^k e^{-i\{(\xi'+i\eta')x'+x_n s(\xi'+i\eta')\}} = e^{-i\{(\xi'+i\eta')x'+x_n s(\xi'+i\eta')\}}.$$

Denoting by L^{*k} the adjoint of L^k, we have that there exists a positive constant M such that

$$|L^{*k}\phi(\xi')| \leqq M|x|^{-k}\sum_{|\alpha|\leqq k}|D_{\xi'}^\alpha\phi(\xi')|, \qquad \eta'\in V\cup\{0\},\quad x\in\{x\in\tilde{\Gamma};\ |x|\geqq 1\}.$$

So noting that $|\exp[-i\{(\xi'+i\eta')x'+x_n s(\xi'+i\eta')\}]|\leqq 1$ for $x\in\tilde{\Gamma}$ and $\xi'+i\eta'\in E+i(V\cup\{0\})$, we obtain by partial integration that $\Phi_{\eta'}(x)$ is rapidly decreasing in $\tilde{\Gamma}$ and that for any multi-index α and positive integer k

$$\lim_{\substack{\eta'\to 0\\ \eta'\in V}}\left\{\sup_{x\in\tilde{\Gamma}\cup\{|x|\leqq 1\}}|(1+|x|)^k D_x^\alpha(\Phi_{\eta'}(x)-\Phi(x))|\right\} = 0. \tag{2.4}$$

Since $\operatorname{supp} f(x)\subset\Gamma$ and $f(x)\in\mathcal{S}'$, we have by (2.4) that

$$\lim_{\substack{\eta'\to 0\\ \eta'\in V}}\int(-\partial/\partial\zeta_n)^\nu\hat{f}(\xi'+i\eta', s(\xi'+i\eta'))\phi(\xi')d\xi' \tag{2.5}$$

$$= \lim_{\substack{\eta'\to 0\\ \eta'\in V}}\langle(ix_n)^\nu f(x), \Phi_{\eta'}(x)\rangle = \langle(ix_n)^\nu f(x), \Phi(x)\rangle.$$

On the other hand, choosing E close to ξ^{jk} and $\rho \in C_0^\infty(\mathbf{R}^1)$ such that $\rho = 1$ in a neighborhood of 0, we set

$$I_R(x) = (ix_n)^\nu \Phi(x)\rho(x/R),$$

$$g_R(\xi) = \phi(\xi')R^{\nu+1}((\partial/\partial\xi_n)^\nu \mathcal{F}^{-1}[\rho])(R(\xi_n - s(\xi')),$$

where $\mathcal{F}^{-1}[\rho] = (2\pi)^{-1}\int \exp\{ix_n\xi_n\}\rho(x_n)dx_n$. We have that the Fourier transform of $g_R(\xi)$ is $I_R(x)$ and that $I_R(x)$ converges to $(ix_n)^\nu\Phi(x)$ in the sense similar to (2.4). Since $f \in \mathcal{S}'$ and $\operatorname{supp} f \subset \Gamma$, it follows that

$$\lim_{R\to\infty} \langle \hat{f}(\xi), g_R(\xi)\rangle = \lim_{R\to\infty} \langle f(x), I_R(x)\rangle = \langle (ix_n)^\nu f(x), \Phi(x)\rangle. \tag{2.6}$$

Since $f = P(D)u$ and u satisfies (1.3), we obtain

$$\lim_{R\to\infty} \langle \hat{f}(\xi), g_R(\xi)\rangle = 0 \tag{2.7}$$

(see [4, proof of theorem 3.1]). The desired equality (2.3) now follows from (2.5), (2.6) and (2.7).

Let K be a compact subcone of $-\Gamma'$ and γ be a multi-index such that $P^{(\gamma)}(\zeta)$ is a non-zero constant. Since the intersection of K and the unit ball is contained in $-\Gamma'$, we can choose $\delta > 0$ so that $\xi + i\eta + D(\delta|\eta|) \subset \mathbf{R}^n - i\Gamma'$ for any $\eta \in K$, where $D(\delta|\eta|) = \{\zeta \in \mathbf{C}^n;\ |\zeta_j| \leqq \delta|\eta|,\ j = 1, \cdots, n\}$. Let $\Psi(\zeta)$ be a non-negative integrable function with compact support in $D(\delta|\eta|)$ and $\int \Psi(\zeta)d\zeta = 1$, depending only on $|\zeta_1|, \cdots, |\zeta_n|$. It follows from Propositions 2.2 and 2.3 that there exist positive constants α, β and $M(K)$ such that α, β are independent of K and

$$|\hat{f}(\xi + i\eta)/P(\xi + i\eta)| \cdot \int |\zeta^\gamma|\Psi(\zeta)d\zeta \leqq M_{m,|\gamma|} \int |\hat{f}(\xi + i\eta + \zeta)|\Psi(\zeta)d\zeta$$

$$\leqq M(K)(1 + |\xi + i\eta|)^\beta(1 + |\eta|^{-\alpha}), \qquad \xi + i\eta \in \mathbf{R}^n + iK.$$

On the other hand, we have for some ε with $0 < \varepsilon < \delta$

$$\int |\zeta^\gamma|\Psi(\zeta)d\zeta \geqq \left(\inf_{\substack{|\zeta_j| > \varepsilon|\eta| \\ j=1,\cdots,n}} |\zeta^\gamma|\right) \cdot \int \Psi(\zeta)d\zeta = (\varepsilon|\eta|)^{|\gamma|}.$$

So we obtain with another constant $M(K)$ depending only on K that

$$|\hat{f}(\xi + i\eta)/P(\xi + i\eta)| \leqq M(K)(1 + |\xi + i\eta|)^\beta(1 + |\eta|^{-(\alpha+m)})$$

for any $\xi + i\eta \in \mathbf{R}^n + iK$. In virtue of Proposition 2.2, it follows from this and the convexity of Γ that there exists a tempered distribution g such that $\operatorname{supp} g \subset \Gamma$ and $\hat{g}(\xi) = \lim_{\eta \to 0\, \eta \in -\Gamma'} \hat{f}(\xi + i\eta)/P(\xi + i\eta)$.

Summing up, we have proved

THEOREM 2.4. *Let Γ be a closed convex proper cone and P satisfies the hypotheses of Theorem 1. Assume that the open cone C in $\mathbf{R}^n - \Gamma_y$ has the properties* (i) *in Theorem* 1. *Let $u \in \mathcal{S}' \cap L^2_{\mathrm{loc}}(\mathbf{R}^n - \Gamma_y)$ and assume that* $\operatorname{supp} P(D)u \subset \Gamma_y$. *If*

$$\varliminf_{R \to \infty} R^{-1} \int_{C_R} |u(x)|^2 dx = 0,$$

then it follows that $P(D)v = P(D)u$ for some $v \in \mathcal{S}'$ with $\operatorname{supp} v \subset \Gamma_y$.

In virtue of the theorem due to Hörmander [4, theorem 2.4], Theorem 1 follows from Theorem 2.4 and the fact that Γ is a proper cone in $\mathbf{R}^n$.

3. Remarks. Examples

We first examine the assumptions of Theorem 1. The following two theorems can be proved in the same way as in the theorems due to Hörmander [4, theorems 3.4 and 3.6].

THEOREM 3.1. *Let Γ be a closed convex proper cone. Assume that P has an irreducible factor p with $\{\zeta \in \mathbf{R}^n - i\Gamma';\, p(\zeta) = 0\} \neq \emptyset$ which is not proportional to a real polynomial or has no simple real zeros. For any integer N one can then find $u \in L^\infty \cap C^\infty$ so that* $\operatorname{supp} P(D)u \subset \Gamma$ *and $u(x) = o(|x|^{-N})$ but* $\operatorname{supp} u \not\subset \Gamma$.

THEOREM 3.2. *Let Γ be the interior of a closed convex proper semi-algebraic cone and p be an irreducible real polynomial with $\{\zeta \in \mathbf{R}^n - i\Gamma';\, p(\zeta) = 0\} \neq \emptyset$. Assume that there is no real ξ with $p(\xi) = 0$ and* $\operatorname{grad} p(\xi) \neq 0$ *such that* $\operatorname{grad} p(\xi)$ *and* $-\operatorname{grad} p(\xi)$ *are both in $\mathbf{R}^n - \Gamma$. Then one can find $u \in \mathcal{S}' \cap C^\infty$ for every integer N such that* $\operatorname{supp} p(D)u \subset \Gamma$ *and $u(x) = o(|x|^{-N})$ in $\mathbf{R}^n - \Gamma$ but* $\operatorname{supp} u \not\subset \Gamma$.

The following theorem, which is well-known (see [4], [8, 9]), says that the estimate (1.3) of Theorem 1 is best possible.

THEOREM 3.3. *Assume that the set B_j given in Theorem* 1 *is non-empty. Let C be an open cone in $\mathbf{R}^n$ such that it contains* $\operatorname{grad} P_j(\xi)$ *or* $-\operatorname{grad} P_j(\xi)$ *for some $\xi \in B_j$. Then there exists a C^∞-function u such that $P(D)u = 0$ in $\mathbf{R}^n$ and* $\varliminf_{R \to \infty} R^{-1} \int_{C_R} |u(x)|^2 dx > 0$, $C_R = \{x \in C;\ R < |x| < 2R\}$.

The following example shows that it is necessary to assume that $\overline{A_j^k} \cap B_j \neq \emptyset$.

EXAMPLE 3.4. Let $P(D)=(-\Delta+1)^2-D_n$ and $\Gamma=\{x\in\mathbf{R}^n;\ -x_n\geqq|x'|\}$ where $x'=(x_1,\cdots,x_{n-1})$. Then for every integer N one can find $u\in\mathcal{S}'\cap C^\infty$ such that $\operatorname{supp} P(D)u\subset\Gamma$ and $u(x)=o(|x|^{-N})$ in $\mathbf{R}^n-\Gamma$ but $\operatorname{supp} u\not\subset\Gamma$. To see this, we note that $-\Gamma'=\{\eta\in\mathbf{R}^n;\ \eta_n>|\eta'|,\ \eta=(\eta',\eta_n)\}$ and that

$$P(\zeta)=\left(\sum_{j=1}^n \zeta_j^2+1-\sqrt{\zeta_n}\right)\left(\sum_{j=1}^n \zeta_j^2+1+\sqrt{\zeta_n}\right)\equiv P_1(\zeta)P_2(\zeta),\qquad \zeta\in\mathbf{R}^n-i\Gamma',$$

where $P_j(\zeta)$ $(j=1,2)$ are holomorphic in $\mathbf{R}^n-i\Gamma'$. We can find ζ^0 in $\{\zeta\in\mathbf{R}^n-i\Gamma';\ P_2(\zeta)=0\}$. Choose $f\in\mathcal{S}$ so that $\operatorname{supp} f\subset\Gamma$ and $\hat{f}(\zeta^0)\neq 0$, and set

$$\hat{u}(\xi)=\xi_n^N\cdot\hat{f}(\xi)(|\xi|^2+1+\sqrt{\xi_n+i0})^{-1},\qquad \xi\in\mathbf{R}^n.$$

Then the inverse Fourier transform of $\hat{u}(\xi)$ satisfies the assertion, since $\xi_n^N(|\xi|^2+1+\sqrt{\xi_n+i0})^{-1}$ is an N-times continuously differential function in $\mathbf{R}^n$.

Next we give two examples in order to illustrate the scope of Theorem 1.

EXAMPLE 3.5. Let $P(D)=\Sigma_{j=1}^n D_j^2-1$ and Γ be any closed proper convex cone with its vertex at the origin. Then we shall show that all assumptions in Theorem 1 concerning P and Γ are satisfied. Let $\xi^0+i\eta^0\in\mathbf{R}^n-i\Gamma'$ satisfy the equation $P(\xi^0+i\eta^0)=0$. Put

$$\gamma(t)=\sqrt{1+t|\eta^0|^2}\cdot\xi^0/|\xi^0|+it\eta^0.$$

Then we see that $\gamma(t)\in\{\zeta\in\mathbf{R}^n-i\Gamma';\ P(\zeta)=0\}$. Since $\xi^0\cdot\eta^0=0$, $\xi^0\notin\Gamma\cup(-\Gamma)$. So we have $P(\xi^0/|\xi^0|)=0$ and $\operatorname{grad} P(\xi^0/|\xi^0|)=2\xi^0/|\xi^0|\notin\Gamma\cup(-\Gamma)$. This completes the proof.

EXAMPLE 3.6. Let $P(D)=D_1^2-\Sigma_{j=2}^n D_j$ and Γ be any closed proper convex cone with its vertex at the origin. Then we assert that all assumptions in Theorem 1 concerning P and Γ are satisfied. When $n=2$, this assertion follows easily. Assume that $n\geqq 3$. Let C be a light cone, that is, $C=\{\eta\in\mathbf{R}^n;\ \eta_1^2>\eta_2^2+\cdots+\eta_n^2\}$. It follows from the hyperbolicity of $P(\xi)$ that $P(\zeta)\neq 0$ when $\zeta\in\mathbf{R}^n+iC$. Thus the assertion follows easily when $-\Gamma'\subset C$. Assume that $-\Gamma'\not\subset C$. Let $\xi^0+i\eta^0\in\mathbf{R}^n+i\{(-\Gamma')-C\}$ satisfy the equation $P(\xi^0+i\eta^0)=0$. Without loss of generality we may assume that $\xi_1^0>0$. It follows from the homogeneity of $P(\xi)$ that $P(\varepsilon\xi^0+i\varepsilon\eta^0)=0$ for any positive ε. Denote by $T(\varepsilon\eta^0)$ the plane in $\mathbf{R}^n$ which is perpendicular to $(\varepsilon\eta_1^0,-\varepsilon\eta_2^0,\cdots,-\varepsilon\eta_n^0)$ through the origin. Then the intersection $I(\varepsilon\eta^0)$ of $T(\varepsilon\eta^0)$ and the hypersurface

$$\{\xi\in\mathbf{R}^n;\ \xi_1^2-\xi_2^2-\cdots-\xi_n^2=(\varepsilon\eta_1^0)^2-(\varepsilon\eta_2^0)^2-\cdots-(\varepsilon\eta_n^0)^2\}$$

is not empty. In fact, it is easily seen that $\varepsilon\xi^0 \in I(\varepsilon\eta^0)$. Therefore, let us denote by $S(\varepsilon\eta^0)$ the connected component of $I(\varepsilon\eta^0)$ which contains the point $\varepsilon\xi^0$.

On the other hand, since $\varepsilon\eta^0 \in (-\Gamma') - C$ and $n \geqq 3$, there exists a point $\xi^1 \in T(\varepsilon\xi^0)^{\check{}}$ such that $\xi^1_1 > 0$ and $P(\xi^1) = 0$. Note that $\operatorname{grad} P(\xi^1) = 2(\xi^1_1, -\xi^1_2, \cdots, -\xi^1_n)$. Since $(\xi^1_1, -\xi^1_2, \cdots, -\xi^1_n) \cdot (\varepsilon\eta^0_1, \varepsilon\eta^0_2, \cdots, \varepsilon\eta^0_n) = 0$ and $\varepsilon\eta^0 \in -\Gamma'$, we have $\operatorname{grad} P(\xi^1) \not\in \Gamma \cup (-\Gamma)$. Without loss of generality, we may assume that the root of P is of the form

$$\zeta_1 = s(\zeta'), \qquad \zeta' = (\zeta_2, \cdots, \zeta_n)$$

in a small complex neighborhood U of ξ^1, where s is a holomorphic function in an open set $\Omega \subset \mathbf{C}^{n-1}$, $s(\xi')$ is real valued for $\xi' \in \Omega \cap \mathbf{R}^{n-1}$, $\xi^1 = (\xi^1_1, s((\xi^1)'))$, and $(\xi^1_1, -\xi^1_2, \cdots, -\xi^1_n)$ is proportional to $(-1, \operatorname{grad} s((\xi^1)'))$. By Taylor expansion, we have

$$\operatorname{Im} s(\xi' + i\eta') = \operatorname{grad} s(\xi') \cdot \eta' + O(|\eta'|^2)$$

for small $|\eta'|$. Thus we can find $\xi' \in \mathbf{R}^{n-1}$ such that

$$\xi' + i\varepsilon(\eta^0)' \in \Omega, \qquad \operatorname{Im} s(\xi' + i\varepsilon(\eta^0)') = \varepsilon\eta^0_1$$

for a sufficiently small number ε. This means that

$$(s(\xi' + i\varepsilon(\eta^0)'), \xi' + i\varepsilon(\eta^0)') \in U \cap S(\varepsilon\eta^0) \times i\{\varepsilon\eta^0\}.$$

So there exists a path $\gamma(t)$ ($t \in [0,1]$) such that $\gamma(t) \in \{\zeta \in \mathbf{R}^n - i\Gamma'; P(\zeta) = 0\}$, $\gamma(0) = \xi^1$ and $\gamma(1) = \varepsilon\xi^0 + i\varepsilon\eta^0$.

Finally, we state that Theorem 1 can be generalized to more general equations $(P(D) + \sum_{j=1}^N q_j(x)Q_j(D))u = f$, where $|q_j(x)| \leqq Me^{-a|x|}$ for some M and $a > 0$, and $\sup_{\zeta \in \mathbf{R}^n - i\Gamma'} |Q_j(\zeta)| (\sum_{|\alpha| \geqq 0} |P^{(\alpha)}(\zeta)|^2)^{-1/2} < \infty$.

THEOREM 2. *Let Γ be a convex proper closed cone in $\mathbf{R}^n$ with its vertex at the origin and $P(\zeta) = Q(\zeta)\prod_{j=1}^p (P_j(\zeta))^{m_j}$, where $Q(\zeta)$ is a polynomial with complex coefficients such that $Q(\zeta) \neq 0$ for any $\mathbf{R}^n - i\Gamma'$ and $P_j(\zeta)$ are irreducible polynomials with real coefficients. Let G_j be a non-empty set such that*

$$G_j \subset \{\xi \in \mathbf{R}^n;\ P_j(\xi) = 0, \operatorname{grad} P_j(\xi) \neq 0, \operatorname{grad} P_j(\xi) \not\in \Gamma \cup (-\Gamma)\}.$$

Assume that for any $j = 1, \cdots, p$ and $r > 0$ there exist constants a_k ($k = 0, \cdots, l$) such that (i) $a_0 = 0$, $a_l > r$, $0 < a_k - a_{k-1} < a$ $(k = 1, \cdots, l)$; (ii) *for any $\zeta^0 \in \{\zeta \in \mathbf{R}^n - i\Gamma';\ P_j(\zeta) = 0, |\operatorname{Im}\zeta| < a_k\}$ ($k = 1, \cdots, l$) there exists a continuous path $\gamma(t)$ ($t \in [0,1]$) with $\gamma(0) = \zeta^0$, $\gamma(1) \in G_j$ and*

$$\gamma(t) \in \{\zeta \in \mathbf{R}^n - i\Gamma';\ |\operatorname{Im}\zeta| < a_k,\ P_j(\zeta) = 0, \operatorname{grad} P_j(\zeta) \neq 0\}, \qquad 0 < t < 1.$$

Let y be a point in $\mathbf{R}^n$. Let *C be an open cone in $\mathbf{R}^n - \Gamma_y$ such that* (i)

$\bigcup_{j=1}^{p} \{\pm \operatorname{grad} P_j(\xi);\ \xi \in G_j\} \subset C$; (ii) *for every real analytic manifold* $M \subset \{\xi \in \mathbf{R}^n;\ P(\xi) = 0\}$ *and* $\xi^0 \in M$, *C contains some normal of M at* ξ^0. *Set*

$$C_R = \{x \in C;\ R < |x| < 2R\}.$$

Let $u \in \mathscr{S}' \cap L^2_{\mathrm{loc}}(\mathbf{R}^n)$, *and for some* ν

$$\int_{|x| \leqq R} |Q_j(D)u|^2 dx \leqq MR^\nu, \qquad R > 0, \quad j = 1, \cdots, N.$$

Suppose that u is a solution of the equation

$$\left(P(D) + \sum_{j=1}^{N} q_j(x)Q_j(D)\right) u = f, \qquad \operatorname{supp} f \subset \Gamma_y, \qquad f \in L^2(\mathbf{R}^n), \tag{3.1}$$

where $|q_j(x)| \leqq Me^{-a|x|}$ *and* $\sup_{\zeta \in \mathbf{R}^n - i\Gamma'} |Q_j(\zeta)| (\sum_{|\alpha| \geqq 0} |P^{(\alpha)}(\zeta)|^2)^{-1/2} < \infty$. *If u satisfies the condition*

$$\lim_{R \to \infty} R^{-1} \int_{C_R} |u(x)|^2 dx = 0, \tag{3.2}$$

then $\operatorname{supp} u \subset \Gamma_{y'}$ *for some* y'.

We leave the proof to the reader (cf. [9, theorem 5]).

References

1. S. Agmon, *Lower bounds for solutions of Schrödinger type equations in unbounded domains*, Proc. Internat. Conf. on Functional Analysis and Related Topics, Tokyo, 1969, pp. 216–224.
2. S. Agmon, *Spectral properties of Schrödinger operators*, Actes Congr. Internat. Math., Vol. 2, 1970, pp. 679–683.
3. L. Hörmander, *Linear partial differential operators*, Springer Verlag, Berlin–Göttingen–Heidelberg, 1963.
4. L. Hörmander, *Lower bounds at infinity for solutions of differential equations with constant coefficients*, Israel J. Math. **16** (1973), 103–116.
5. T. Kato, *Growth properties of solutions of the reduce wave equation with a variable coefficient*, Comm. Pure Appl. Math. **12** (1959), 403–425.
6. R. Konno, *Non-existence of positive eigenvalues of Schrödinger operators in infinite domains*, J. Fac. Sci. Univ. Tokyo, Sect. I A Math. **19** (1972), 393–402.
7. W. Littman, *Decay at infinity of solutions to partial differential equations; removal of the curvature assumption*, Israel J. Math. **8** (1970), 403–407.
8. M. Murata, *A theorem of Liouville type for partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. I A Math. **21** (1974), 395–404.
9. M. Murata, *Asymptotic behaviors at infinity of solutions of certain linear partial differential equations*, J. Fac. Sci. Univ. Tokyo, Sect. I A Math. **23** (1976), 107–148.
10. F. Rellich, *Über das asymptotische Verhalten der Lösunger von* $\Delta u + k^2 u = 0$ *in unendlichen Gebieten*, J. Ber. Dt. Math. Ver. **53** (1943), 57–65.
11. W. Rudin, *Lectures on the edge-of-the-wedge theorem*, CBMS Regional Conf., No. **6**, 1970.
12. Y. Shibata, *Liouville type theorem for a system* $\{P(D), B_j(D),\ j = 1, \cdots, p\}$ *of differential operators with constant coefficients in a half space*, to appear.

13. T. Tayoshi, *The asymptotic behavior of the solutions of* $(\Delta + \lambda)u = 0$ *in a domain with the unbounded boundary*, Publ. Res. Inst. Math. Sci. **8** (1972/73), 375–391.

14. F. Trèves, *Differential polynomials and decay at infinity*, Bull. Amer. Math. Soc. **66** (1960), 183–186.

15. F. Trèves, *Lectures on partial differential equations with constant coefficients*, Notas de Matemática No. **27**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1961.

16. V. S. Vladimirov, *Methods of the theory of functions of several complex variables*, "Nauka", Moscow, 1964; English transl., M.I.T. Press, Cambridge, Mass., 1966.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOKYO METROPOLITAN UNIVERSITY
FUKAZAWA-CHO, SETAGAYA-KU, TOKYO, 158 JAPAN

AND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TSUKUBA
SAKURA-MURA NIIHARI-GUN IBARAKI, 300–31 JAPAN